Distributed Knowledge on 'Knowing whether'

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Abstract. The classical distributed knowledge is based on the traditional understanding of 'knowing that something', but there may be some differences considering other kind of 'knowing'. Apart form 'knowing that', the concept of 'knowing whether', 'knowing how' and 'knowing why' have been put forward for some time and relevant work is ongoing. In this paper, we try to study the distributed knowledge on 'knowing whether', expecting to find some differences compared with the classical one.

Keywords: distributed knowledge, knowing whether, pseudo-model, completeness

1 Introduction

The concept of distributed knowledge was originally put forward in the field of computer science when discussing the distributed system. Generally, the distributed knowledge is characterized by intersection of information states. Surely the information of different agents is presented in the way of 'knowing that', classically. However, 'knowing whether' is something new put forward these years by Prof. Wang. We've found there are some special features which 'knowing that' does not have. Thus, this inspires us to consider the combination of these two concept. That means talking about the distributed knowledge in a new viewpoint.

1.1 Classical distributed knowledge

Given a set of agents A, and a set of propositional variable P, the language L^D of epistemic logic with distributed knowledge is given by following definition:

$$\Phi := p \mid \phi \wedge \psi \mid \neg\phi \mid K_i\phi \mid D\phi$$

where $p \in P$, $i \in A$.

Definition 1. *The definitions of satisfaction of sentences of L in a model M are as usual for the atomic formulas and the Boolean formulas. Now we just consider the nontrivial cases:*

$$M, w \models K_i\phi \ iff \ for \ all \ v \ such \ that \ w \to_i v: \ M, v \models \phi \quad (1)$$

$$M, w \models D\phi \ iff \ for \ all \ v \ such \ that \ w \to_i v \ for \ each \ i \in A: \ M, v \models \phi \quad (2)$$

As we have seen, the semantic definition of the $K_i\phi$ is just like the classical modal operator. And the semantic of the $D\phi$ is the intersection of the information from every agent $i{\in}A$.

Based on the $K_i\phi$, classical logic about distributed knowledge gave its Axioms System as follows:

Definition 2. *Axiom System:*

$$TAUT\ and\ all\ instances\ of\ tautologies \tag{3}$$

$$K_i\phi{\wedge}K_i(\phi{\rightarrow}\psi){\rightarrow}K_i\psi \tag{4}$$

$$K_i\phi{\leftrightarrow}D\phi\ when\ A = \{i\} \tag{5}$$

$$K_i\phi{\rightarrow}D\phi \tag{6}$$

$$D(\phi{\rightarrow}\psi){\wedge}D\phi{\rightarrow}D\psi \tag{7}$$

$$MP : \phi, \phi \rightarrow \psi\ /\psi \tag{8}$$

$$NEC:\ from\ \phi\ infer\ K_i\phi \tag{9}$$

Having introduced the traditional definition of the distributed knowledge, we have to clarify another part of our work, namely epistemic logic about the 'knowing whether'.

1.2 Knowing whether

The work on the 'knowing whether' is based on the concept of noncontingency, which means it is necessarily true or it is necessarily false. Intuitively, someone knows whether ϕ means he definitely knows that ϕ is true or he definitely knows ϕ is false. This concept is often sufficient to express interesting propositions without using the more expressive 'knowing that' construction. Firstly, we should introduce the semantics of the noncontingency operator:

Definition 3. *The definition of the satisfaction of the sentences in form of $\triangle_i\phi$, where $\triangle_i\phi$ means agent i knows whether ϕ intuitively.*

$$M, s \models\triangle_i\phi\ iff\ for\ all\ t_1, t_2\ such\ that\ s{\rightarrow_i}t_1, s{\rightarrow_i}t_2,\ :\ (M, t_1 \models\phi \Leftrightarrow M, t_2 \models\phi) \tag{10}$$

More clearly, $M, s \models\triangle_i\phi\ iff\ M, s \models K_i\phi\ or\ M, s{\models}K_i\neg\phi$

Actually it is quite different from 'knowing that'. Comparing the axioms of both systems, K-axiom is not satisfied in NCL(the axiom system about 'knowing whether') in which two new fundamental axioms replace the role of K-axiom played in traditional epistemic logic.

Definition 4. *Axiom System for NCL:*

$$TAUT\ and\ all\ instances\ of\ tautologies$$

$$\triangle_i\phi \leftrightarrow \triangle_i\neg\phi \tag{11}$$

$$(\triangle Con)\ \triangle_i(\chi \to \phi) \wedge \triangle_i(\neg\chi \to \phi) \to \triangle_i\phi \tag{12}$$

$$(\triangle Dis)\ \triangle_i\phi \to \triangle_i(\phi \to \psi) \vee \triangle_i(\neg\phi \to \chi) \tag{13}$$

$$MP : \phi, \phi \to \psi\ /\psi$$

$$NEC :\ from\ \phi\ infer\ \triangle_i\phi \tag{14}$$

$$RE\triangle :\ from\ \phi \leftrightarrow \psi\ infer\ \triangle_i\phi{\leftrightarrow}\triangle_i\psi \tag{15}$$

In NCL, $(\triangle Con)$ and $(\triangle Dis)$ are the fundamental axioms in this proof system. Moreover, here is no SUB-rule. Since the proof of SUB is in use of K-axiom which is lacking in NCL, SUB can't be given directly. But adding the rule $RE\triangle$, SUB is admissible in NCL.

As we have mentioned above, distributed knowledge is based on the most fundamental epistemic logic. However, NCL is not a normal modal logic, which forces us to reconstruct the system about distributed knowledge on 'knowing whether'.

2 Syntax and Semantics of DNCL

In this section, we first define a logical language including noncontingency and distribution operators.

Definition 5. *Let a set P of propositional variables and a finite set A of agents i. The logical language of DNCL is defined as:*

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi{\wedge}\phi \mid \triangle_i\phi \mid D^{\triangle}\phi \tag{16}$$

We use the operator $D^{\triangle}$ to refer to the distributed knowledge on 'knowing whether' which differs from the classical one, D.

Given the language, we will define the model and semantics in a standard way.

2.1 Models and semantics

Before we give the definition of semantics, we should clarify the models we would use in following part.

Definition 6. *A model is triple $M = \langle S, \{\to_i | i{\in}A\}, V\rangle$ where S is a nonempty set of possible worlds, $\to_i$ is a binary relation over S for each $i{\in}A$, and V is a valuation function assigning a set of worlds $V(p){\subseteq} S$ to each $p{\in}P$. Given a world $s{\in}S$, the pair (M,s) is a pointed model. A frame is a pair $F = \langle S, \{\to_i | i{\in}A\}\rangle$, just equals to the model without a valuation.*

Now we will give the first important definition, semantics.

Definition 7. *Given a model $M = \langle S, \{\rightarrow_i | i \in A\}, V \rangle$, the semantics of DNCL is defined as follows:*

$$M, s \models \top \ always$$

$$M, s \models p \ iff \ s \in V(p)$$

$$M, s \models \neg\phi \ iff \ not \ M, s \models \phi$$

$$M, s \models \triangle_i \phi \ iff \ for \ all \ t_1, t_2 \ such \ that \ s \rightarrow_i t_1, s \rightarrow_i t_2, : \ (M, t_1 \models \phi \Leftrightarrow M, t_2 \models \phi)$$

$$M, s \models D^{\triangle}\phi \ iff \ for \ all \ t \ such \ that \ s \rightarrow_i t \ for \ each \ i, \ it \ holds \ that M, t \models \phi$$

$$or \ for \ all \ t \ such \ that \ s \rightarrow_i t \ for \ each \ i, \ it \ holds \ that \ M, t \models \neg\phi \tag{17}$$

Considering the satisfaction condition of $D^{\triangle}\phi$, it also describes the intersection of information from all agents. But once each agent has the same value of ϕ, we can conclude that 'knowing whether ϕ' is the distributed knowledge over the group A.

If we realize the features of 'knowing whether' and the new distributed knowledge, we will construct the axiom system in a new way. And the main idea depends on NCL.

2.2 Axiomatization

In this section we give a complete Hilbert-style proof system for the logic DNCL on the class of all frames.

Definition 8. *The proof system DNCL consists of the following axiom schemas and inference rules:*

$$TAUT \ and \ all \ instances \ of \ tautologies$$

$$\triangle_i \phi \leftrightarrow \triangle_i \neg\phi$$

$$(\triangle Con) \ \triangle_i(\chi \rightarrow \phi) \wedge \triangle_i(\neg\chi \rightarrow \phi) \rightarrow \triangle_i \phi$$

$$(\triangle Dis) \ \triangle_i \phi \rightarrow \triangle_i(\phi \rightarrow \psi) \vee \triangle_i(\neg\phi \rightarrow \chi)$$

$$\triangle_i \phi \leftrightarrow D^{\triangle}\phi \ when \ A = \{i\} \tag{18}$$

$$\triangle_i \phi \rightarrow D^{\triangle}\phi \tag{19}$$

$$(D^{\triangle}Con) \ D^{\triangle}(\chi \rightarrow \phi) \wedge D^{\triangle}(\neg\chi \rightarrow \phi) \rightarrow D^{\triangle}\phi \tag{20}$$

$$(D^{\triangle}Dis) \ D^{\triangle}\phi \rightarrow D^{\triangle}(\phi \rightarrow \psi) \vee D^{\triangle}(\neg\phi \rightarrow \chi) \tag{21}$$

$$MP : \phi, \phi \rightarrow \psi \ / \psi$$

$$NEC : \ from \ \phi \ infer \ \triangle_i \phi$$

$$RE\triangle : \ from \ \phi \leftrightarrow \psi \ infer \ \triangle_i \phi \leftrightarrow \triangle_i \psi$$

$$RED^{\triangle} : \ from \ \phi \leftrightarrow \psi \ infer \ D^{\triangle}\phi \leftrightarrow D^{\triangle}\psi \tag{22}$$

A derivation of DNCL is a finite sequence of DNCL-formulas such that each formulas is either the instaniation of an axiom or the result of applying an inference rule to prior formulas in the sequence. A formula is $\phi \in Th(DNCL)$ is called provable, or a theorem, notation $\vdash \phi$, if it occurs in a derivation of DNCL.

We can see that in this axiom system, it remains the basic axioms and rules from NCL. Besides, we add some new axioms with the operator $D^{\triangle}$ while in the same form with ($\triangle Con$), ($\triangle Dis$) and $RE\triangle$, namely ($D^{\triangle}Con$), ($D^{\triangle}Dis$) and $RED^{\triangle}$.

Please pay much attention to the axiom(19) because it constructs a bridge between the operator $\triangle$ and $D^{\triangle}$, which is quite useful in the proof of Completeness. Then we should show the soundness of DNCL.

Proposition 1. *The proof system DNCL is sound with respect to the class of all frames.*

Proof: Actually we just need to prove the soundness of the axioms(18), (19), (20), (21) and the rule(22). Because other axioms and rules have been proved in Prof.Wang's paper on 'knowing whether'. Among these axioms, the proof of ($D^{\triangle}Con$), ($D^{\triangle}Dis$) and $RED^{\triangle}$ are just similar to the proof of ($\triangle Con$), ($\triangle Dis$) and $RE\triangle$. We just consider the case where $s{\rightarrow_i}t$ for each $i{\in}A$ instead of $s{\rightarrow_i}t$ for some $i{\in}A$.

Obviously, (18) sound. Because if there is only one agent in A, the distributed knowledge is just same to the single agent's knowledge. The intersection of information is just equals to the set of information of the agent i.

As for axiom(19), let $M, S{\models}\triangle_i\phi$, if and only if for all t with $s{\rightarrow_i}t$, such that $M, t{\models}\phi$ or for all t with $s{\rightarrow_i}t$, such that $M, t{\models}\neg\phi$. If there is no u with $s{\rightarrow_i}u$ for each $i{\in}A$, then $M, S{\models}D^{\triangle}\phi$ holds trivially. If there is u with $s{\rightarrow_i}u$ for each $i{\in}A$, then there must be $M, u{\models}\phi$ or $M, u{\models}\neg\phi$ since $s{\rightarrow_i}u$. Thus, we have $M, S{\models}D^{\triangle}\phi$.

Therefore we have proved the soundness of DNCL.

3 Completeness to the Class of $\mathcal{K}$-frames

In this section, we are going to prove the completeness of DNCL. Generally, the proof of completeness is complicated and logicians have developed a whole processes of the proof on completeness of modal logic. The most standard method is to use the 'canonical model'. However the classical way can not be taken immediately due to the characteristics of the operator $D^{\triangle}$.

Actually we will take a method put forward by Fagin et al. (1992) given for DS5 which is effective in dealing with the problems arose by the operator. It's main idea is introducing a new 'kit', named 'pseudo-models'. Briefly, we will show that DNCL is 'pseudo-satisfied' in 'pseudo-model'. And then we can transform the pseudo-model into a real one to prove that DNCL can also be satisfied in the real model in help with the 'pseudo-satisfied'.

Now, I will show the proof in detailed.

3.1 Construct the pseudo-model

The main idea of the pseudo-model is just regard the operator $D^{\triangle}$ as a new common operator, same to $\triangle_i$. So we use a new relation $\rightarrow_D$ to refer to the accessibility in the pseudo-model.

Definition 9. *Let* $M^* = \langle S,\ \{\rightarrow_i^* | i \in A\},\ \rightarrow_D, V\rangle$ *is a pseudo-model, in which*

$$S = \{s | s\ is\ a\ maximal\ set\ of\ DNCL\}$$

$$s_1 \rightarrow_i^* s_2\ iff\ there\ exists\ \chi\ such\ that$$

$$1.\ \neg\triangle_i\chi \in s_1\ and$$

$$2.\ for\ all\ \phi:\ \triangle_i\phi \wedge \triangle_i(\chi \rightarrow \phi) \in s_1,\ implies\ \phi \in s_2$$

$$s_1 \rightarrow_D s_2\ iff\ there\ exists\ \chi\ such\ that$$

$$1.\ \neg D^{\triangle}\chi \in s_1\ and$$

$$2.\ for\ all\ \phi:\ D^{\triangle}\phi \wedge D^{\triangle}(\chi \rightarrow \phi) \in s_1,\ implies\ \phi \in s_2$$

$$V(p) = \{s \in S | p \in s\}$$

The new satisfaction is 'pseudo-satisfy': $M^*, s \models^* \phi$ *is just as ordinary in Kripke models except that* $M^*, s \models^* D^{\triangle}\phi$ *iff for all t with* $s \rightarrow_D t$, $M^*, t \models^* \phi$ *or* $M^*, t \models^* \neg\phi$.

Clearly, $\rightarrow_D$ is just same to the $\rightarrow_i^*$ and from the definition of pseudo-satisfaction, $D^{\triangle}$ is same to $\rightarrow_i$. In a standard way, we would prove the truth lemma in the following.

Lemma 1. *Truth Lemma*: $M^*, s \models^* \phi$ *iff* $\phi \in s$ *for all* $\phi \in L^D$

Proof: Induction on ϕ. *The nontrivial cases are when* $\phi = \triangle_i\psi$ *and when* $\phi = D^{\triangle}\psi$. *But we have regarded* $D^{\triangle}$ *as* $\triangle_i$, *so we can just prove the second case in the absolutely same way as the first case. Since Prof. Wang has proved it in his paper, we just need to refer to that proof.*

3.2 A tree-like model

We have proved that DNCL can be satisfied in M^*. Before we transform it into a real model, we should firstly change it into a tree-like model which would make it convenient to obtain the real model from it.

The method of constructing a tree-like model put forward by Fagin is proper for any model, of course for the pseudo-model. l will sketch the method here:

The approach is to create the states of the tree-like model M_T^* from the original model M^*. The states in M_T^* is at various 'level'. The first level T_1 contains precisely S, the set of states of M^*. Assume inductively that we have defined the set T_k of states at level k. Then, for each $s \in S$, each $v \in T_k$, and each agent i, we define a new, distinct state $z_{s,v,i}$. We may refer to $z_{s,v,i}$ as an i-child of v, and to v as the parent of $z_{s,v,i}$. Let $T = \cup\{T_k | k \geq 1\}$. Define g:$T \rightarrow S$ by letting g(s)=s if $s \in T_1$, and g($z_{s,v,i}$)=s for $z_{s,v,i} \in T_k$ where $k \geq 2$. Intuitively, we shall construct M_T^* with state space T such that the set of formulas Φ on $s \in T$ is same to that on the state $g(s) \in S$.

And then we define the relations in M_T^*. Let $s \rightarrow_i^T t$ iff t is an i-child of s and $g(s) \rightarrow_i^* g(t)$. Define $V^T(s) = V(g(s))$.

Now we obtain the tree-like model $M_T^* = \langle T, \{\rightarrow_i^T | i \in A\}, V^T\rangle$ from the pseudo-model M^*. We will show that the set of formulas Φ on $s \in T$ is same to that on the state $g(s) \in S$.

Proposition 2. $M_T^*, s\models^* \psi$ *iff* $M^*, g(s)\models^* \psi$

Proof: Induction on ψ*. Considering that the case when* $\psi = \triangle_i\phi$ *is same to the case when* $\psi = D^{\triangle}\phi$*. So we can just prove the nontrivial case when* $\psi = \triangle_i\phi$*.*

($\leftarrow$) *Assume first that* $M_T^*, s\models^*$*not* $\triangle_i\phi$*. Thus there are* t_1 *and* t_2 *with* $s\rightarrow_i^T t_1$ *and* $s\rightarrow_i^T t_2$*, such that* $M_T^*, t_1\models^* \phi$ *and* $M_T^*, t_2\models^* \neg\phi$*. According to the definition of* $\rightarrow_i^T$*, there are* $g(s)\rightarrow_i^* g(t_1)$ *and* $g(s)\rightarrow_i^* g(t_2)$*. By inductive hypothesis(abbreviate it to IH in the followings),* $M^*, g(t_1)\models^* \phi$ *and* $M^*, g(t_2)\models^* \neg\phi$*. So we get* $M^*, g(s)\not\models^* \triangle_i\phi$

($\rightarrow$) *Assume that* $M^*, g(s)\models^*$*not* $\triangle_i\phi$*. Thus there are* w_1 *and* w_2 *with* $g(s)\rightarrow_i^* w_1$ *and* $g(s)\rightarrow_i^* w_2$*, such that* $M^*, w_1\models^* \phi$ *and* $M^*, w_2\models^* \neg\phi$*. According to the definition of the state space T, there must be* $z_{w_1,s,i}$ *and* $z_{w_2,s,i}$ *with* $s\rightarrow_i^T z_{w_1,s,i}$ *and* $s\rightarrow_i^T z_{w_2,s,i}$*. By IH,* $M_T^*, z_{w_1,s,i}\models^* \phi$ *and* $M_T^*, z_{w_2,s,i}\models^* \neg\phi$*. So we get* $M_T^*, s\not\models^* n\triangle_i\phi$*.*

We have proved it as desired.

We have constructed a tree-like model from the pseudo-model and actually they have the same set of formulas Φ on corresponding states. However, M_T^* is also a pseudo-model. Thus, the remaining work we have to do is transforming the tree-like model into a real model.

3.3 Construct the real model

The key process to construct the real model is to reconstruct the relations in the tree-like model M_T^*. And the state space and the valuation inherited from M_T^* directly.

Definition 10. *Let* $M = \langle T, \{\rightarrow_i | i\in A\}, V^T\rangle$*. We set* $w\rightarrow_i^+ v$ *if* $w\rightarrow_D v$ *in* M_T^* *for each* $i\in A$*. Let* $\rightarrow_i = \rightarrow_i^T \cup \rightarrow_i^+$ *for any* $i\in A$*.*

Because M and M_T^* has the same state space T, we should show that the set of formulas Φ satisfied in M is exactly the set of formulas pseudo-satisfied in M_T^*.

Proposition 3. *For any* $\psi\in\Phi$*, there is* $M, s\models\psi$ *iff* $M_T^*, s\models^* \psi$*.*

Proof: Induction on ψ*. If* ψ *is a propositional variable or Boolean formula, it's immediate. We just consider the nontrivial case.*

Case 1: If $\psi = \triangle_i\phi$*:*

($\rightarrow$) *Assume* $M_T^*, s\models^*$*not* $\triangle_i\phi$*. Thus, there are* t_1 *and* t_2 *with* $s\rightarrow_i^T t_1$ *and* $s\rightarrow_i^T t_2$*, such that* $M_T^*, t_1\models^* \phi$ *and* $M_T^*, t_2\models^* \neg\phi$*. By IH,* $M, t_1\models\phi$ *and* $M, t_2\models\neg\phi$*, since* $\{\rightarrow_i^T\}\subseteq\{\rightarrow_i | i\in A\}$*. So we get* $M, s\models$*not* $\triangle_i\phi$*.*

($\leftarrow$) *Assume* $M_T^*, s\models^* \triangle_i\phi$*, if and only if for all t with* $s\rightarrow_i^T t$*,such that* $M_T^*, t\models^* \phi$ *or for all t with* $s\rightarrow_i^T t$*,such that* $M_T^*, t\models^* \neg\phi$*. To show* $M, s\models\triangle_i\phi$*, we must show for all t with* $s\rightarrow_i t$ *such that* $M, t\models\phi$ *or for all t with* $s\rightarrow_i t$ *such that* $M, t\models\neg\phi$*. We assume here that* $s\rightarrow_i^T t$ *such that* $M_T^*, t\models^* \phi$*. Here are 2 cases when* $s\rightarrow_i t = s\rightarrow_i^T t$ *or* $s\rightarrow_i u = s\rightarrow_i^+ u$*. If* $s\rightarrow_i t = s\rightarrow_i^T t$*,* $M_T^*, t\models^* \phi$*. By IH,* $M, t\models\phi$*. If* $s\rightarrow_i u = s\rightarrow_i^+ u$*, that means* $s\rightarrow_D u$ *in* M_T^**. Since* $M_T^*, s\models^* \triangle_i\phi$*, according to the axiom(*$\triangle_i\phi\rightarrow D^{\triangle}\phi$*),* $M_T^*, s\models^* D^{\triangle}\phi$*. Now we want to show that* $M_T^*, u\models^* \phi$*. Because* $s\rightarrow_D u$ *in* M_T^**, according to definition of* $\rightarrow_D$*, we know there* *exists* χ *such that* $\neg D^{\triangle}\chi{\in}s$*. We prove by contradiction. Assume* $M_T^*, u{\models}^*\neg\phi$*, by truth lemma,* $\neg\phi{\in}u$*. Since* $\phi{\in}t$*, we have* $\chi{\to}\phi{\in}t$ *for all t with* $s{\to}_i^* t$*. So we have* $\triangle_i(\chi{\to}\phi){\in}s$*. By Axiom,* $D^{\triangle}(\chi{\to}\phi){\in}s$*. Similarly, since* $\neg\phi{\in}u$*, we have* $\chi{\to}\neg\phi{\in}u$ *for all u with* $s{\to}_D u$*. So we have* $D^{\triangle}(\chi{\to}\neg\phi){\in}s$*. By Axiom(*$D^{\triangle}$*con),* $D^{\triangle}(\neg\phi{\to}\neg\chi){\wedge}D^{\triangle}(\phi{\to}\neg\chi){\to}D^{\triangle}\neg\chi$*. So we have* $D^{\triangle}\neg\chi{\in}s$*. That means* $D^{\triangle}\chi{\in}s$*, which contradicts to* $\neg D^{\triangle}\chi{\in}s$*. So* $M_T^*, u{\models}^*\phi$*. By IH,* $M, u{\models}\phi$*. So for all t with* $s{\to}_i t$*, there is* $M, t{\models}\phi$

Assume for all t $s{\to}_i^T t$ *such that* $M_T^*, t{\models}\neg\phi$ *is similar. So for all t with* $s{\to}_i t$ *, there is* $M, t{\models}\phi$ *or for all t with* $s{\to}_i t$ *, there is* $M, t{\models}\neg\phi$*.*

Therefore we get $M, s{\models}\triangle_i\phi$*.*

Case 2: If $\psi = D^{\triangle}\phi$*:*

$(\to)$ *Assume* $M_T^*, s{\models}^*$*not* $D^{\triangle}\phi$*. Thus, there are* t_1 *and* t_2 *with* $s{\to}_D t_1$ *and* $s{\to}_D t_2$*, such that* $M_T^*, t_1{\models}^*\phi$ *and* $M_T^*, t_2{\models}^*\neg\phi$*. By IH,* $M, t_1{\models}\phi$ *and* $M, t_2{\models}\neg\phi$*, since* $s{\to}_D t_1$ *and* $s{\to}_D t_2$*, according to the definition of* $\{{\to}_i^+\}$*, there are* $s{\to}_i t_1$ *and* $s{\to}_i t_2$ *for all* $i{\in}A$*. So we get* $M, s{\not\models}D^{\triangle}\phi$*.*

$(\leftarrow)$ *Assume* $M, s{\not\models}D^{\triangle}\phi$*. Thus, there are* t_1 *and* t_2 *with* $s{\to}_i t_1$ *and* $s{\to}_i t_2$ *for each* $i{\in}A$*, such that* $M, t_1{\models}\phi$ *and* $M, t_2{\models}\neg\phi$*. According to the definition of* $\{{\to}_i\}$ *and* M_T^* *is tree-like, we get to know that* $s{\to}_D t_1$ *and* $s{\to}_D t_2$ *in* M_T^**. By IH, there are* $M_T^*, t_1{\models}^*\phi$ *and* $M_T^*, t_2{\models}^*\neg\phi$*.*

So we have $M_T^*, s{\not\models}^* D^{\triangle}\phi$*.*

Now we have proved the proposition we want.

Theorem 1. *The logic DNCL is complete with respect to the class* $\mathcal{K}$ *of all frames.*

4 Axiomatization: Extension

We have proved the completeness to the class of $\mathcal{K}$-frames above all. As we all know that generally, the epistemic logic on S5-frames would properly describe the knowledge of different agents. So we have to extend our axiom system, say DNCL, to DNCLS5.

As the general way, we add some axioms to DNCL. Because our concept of distributed knowledge is based on 'knowing whether', so we should add the axioms from NCLS5 and construct some new axioms with the operator $D^{\triangle}$ in the same form of these new adding axioms from NCLS5.

Definition 11. *DNCLS5 is obtained by adding following axioms:*

$$(\triangle T)\ \triangle_i\phi{\wedge}\triangle_i(\phi{\to}\psi){\wedge}\phi{\to}\triangle_i\psi \tag{23}$$

$$(w\triangle 5)\ \neg\triangle_i\psi{\to}\triangle_i\neg\triangle_i\phi \tag{24}$$

$$(DT)\ D^{\triangle}\phi{\wedge}D^{\triangle}(\phi{\to}\psi){\wedge}\phi{\to}D^{\triangle}\psi \tag{25}$$

$$(wD5)\ \neg D^{\triangle}\psi{\to}D^{\triangle}\neg D^{\triangle}\phi \tag{26}$$

The proof of the soundness of (DT) and (wD5) is similar to that of ($\triangle T$) and ($w\triangle 5$) which have been given by Prof.Wang in his former work.

Here is something we should pay attention that the formula($\triangle 4$), say $\triangle_i\phi \rightarrow \triangle_i\triangle_i\phi$, can be implied by DNCLS5, which is useful in following proof. In the next section, we shall prove the completeness of DNCLS5.

5 Completeness to the Class of $\mathcal{S}5$-frames

Having the experience in former proof of completeness to $\mathcal{K}$, we can just walk along the process to give the the proof of completeness of DNCLS5.

5.1 A S5-pseudo-model

Similar to the way to construction above, we can construct the new S5-pseudo-model M^* just through adjusting the definition of relations, to let it be a equivalence relation.

Definition 12. *We use the former symbols. Let a S5-pseudo-model* $M^* = \langle S,\ \{\rightarrow_i^* | i \in A\},\ \rightarrow_D, V\rangle$, *where:*

$$s_1 \rightarrow_i^* s_2 \ is\ the\ reflexive\ closure\ of\ the\ relations\ defined\ in\ DNCL.$$

$$s_1 \rightarrow_D s_2 \ is\ the\ reflexive\ closure\ of\ the\ relations\ defined\ in\ DNCL.$$

And the pseudo-satisfaction is defined as above.

And the *truthlemma** is the same in proof of DNCL.

But actually we just construct the pseudo-model here to be a reflexive model. We should show that the $\{\rightarrow_i^* | i \in A\}$ is Euclidean. Since in a pseudo-model, we regard $D^\triangle$ as $\triangle_i$, so the proof is absolutely same to the proof given by Prof.Wang in his paper.

5.2 A S5-tree-like pseudo-model

The method to change M^* into the tree-like model M_T^* is almost same to it mentioned above. But there still be something different here, that is also about the relations.

Definition 13. *We let* $M_T^* = \langle T, \{\rightarrow_i^T | i \in A\}, V^T\rangle$ *be the tree-like pseudo-model again, where* $\{\rightarrow_i^T | i \in A\}$ *is the reflexive, symmetric and transitive closure of the relations we defined above in DNCL. Here much notice we should must put in. We firstly transform the pseudo-model into the tree-like pseudo-model named* M_t^**. Then we do the reflexive closure of* M_t^* *which forms a new model* $M_{tr}^* =< T, R_{tr}^*, V^T >$*. Then we do the symmetric closure of* M_{tr}^* *which forms the model* $M_{trs}^* =< T, R_{trs}^*, V^T >$*. At the last, we do the transitive closure of* M_{trs}^* *and make it into* M_T^**. The state space and the valuation are the same as above.*

It's easy to see that this tree-like pseudo-model M_T^* *is a S5-model.*

And the proof of $M_T^*, s\models^*\psi$ iff $M^*, g(s)\models^*\psi$ is different from the above one. Because we have added more relations in the tree-like model.

Proposition 4. *$M_T^*, s\models^*\psi$ iff $M^*, g(s)\models^*\psi$*

proof: If ψ is a propositional variable, this follows from the fact that $V^T(s) = V(g(s))$. The case where ψ is a Boolean formula is immediate. We now just consider the nontrivial case when $\psi = \triangle_i\phi$.(The case when $\psi = D^\triangle\phi$ is the same)

($\leftarrow$)Assume first that $M_T^*, s\not\models^*\triangle_i\phi$. Thus there are t_1, t_2 with $s\rightarrow_i^T t_1$, $s\rightarrow_i^T t_2$ such that $M_T^*, t_1\models^*\phi$ and $M_T^*, t_2\models^*\neg\phi$. If $s = t_1$ or $s = t_2$, $g(s)\rightarrow_i^* g(t_1)$ or $g(s)\rightarrow_i^* g(t_2)$ holds trivially since m^* is a reflexive model.

If $s \not\models t_1$ and $s \not\models t_2$, we know that before we do the transitive closure, there is a path $P =< s, i, v_1, i, v_2, i, ..., i, v_k, i, t_1 >$ from s to t_1 where every adjacent nodes in it are in the adjacent layers in T. let v_i and v_{i+1} are arbitrary adjacent nodes in P. If $v_i\in T_i$ and $v_{i+1}\in T_{i+1}$, then $(v_i\rightarrow_i^T v_{i+1})\in R_t^*$. By the definition of the pseudo-model, we have $g(v_i)\rightarrow_i^* g(v_{i+1})$.

If $v_i\in T_i$ and $v_{i+1}\in T_{i-1}$, then $(v_i\rightarrow_i^T v_{i+1})\in R_t^* rs$. That means $(v_{i+1}\rightarrow_i^T v_i)\in R_t^*$. By the definition of the pseudo-model, we have $g(v_{i+1})\rightarrow_i^* g(v_i)$. Since $\{\rightarrow_i^*\}$ is symmetric, we have $g(v_i)\rightarrow_i^* g(v_{i+1})$.

According to above, there is a path $W =< g(s), i, g(v_1), i, g(v_{i+1}), i, ..., i, g(v_{i+k}), i, g(t_1) >$ from g(s) to $g(t_1)$. By the transitivity of the $\{\rightarrow_i^*\}$, we have $g(s)\rightarrow_i^* g(t_1)$. By the Inductive hypothesis, we also have $M^*, g(t_1)\models^*\phi$. Similarly, as for t_2, we can get there is $g(s)\rightarrow_i^* g(t_2)$ and $M^*, g(t_2)\models^*\neg\phi$. Thus we have $M^*, g(s)\not\models^*\triangle_i\phi$.

($\rightarrow$) The case is just same to the case in the proof of Proposition 2.

5.3 A real S5-model

The proof of completeness is different from the former because it is involved in many properties of relations, such as reflexive, transitive.

Definition 14. *Let $M = \langle T, \{\rightarrow_i | i\in A\}, V^T\rangle$. We set $w\rightarrow_i^+ v$ if $w\rightarrow_D v$ in M_T^* for each $i\in A$. Let $\rightarrow_i| =$ the transitive closure of $\rightarrow_i^T\cup\rightarrow_i^+$ for any $i\in A$.*

Because $\rightarrow_i^T$ and $\rightarrow_D$ are reflexive and symmetric, it's easy to see that $\{\rightarrow_i | i\in A\}$ is also reflexive and symmetric. That means $\{\rightarrow_i | i\in A\}$ is a equivalence relation. So M is a S5-model. We need to show that:

Proposition 5. *For any $\psi\in\Phi$, there is $M, s\models\psi$ iff $M_t^*, s\models^*\psi$.*

Proof: Induction on ψ. We just consider the nontrivial case.

Case 1: If $\psi = \triangle_i\phi$:

($\rightarrow$) *Assume $M_T^*, s\not\models^*\triangle_i\phi$. Thus, there are t_1 and t_2 with $s\rightarrow_i^T t_1$ and $s\rightarrow_i^T t_2$, such that $M_t^*, t_1\models^*\phi$ and $M_t^*, t_2\models^*\neg\phi$. By IH, $M, t_1\models\phi$ and $M, t_2\models\neg\phi$, since $\rightarrow_i^T\subseteq\rightarrow_i$. So we get $M, s\not\models\triangle_i\phi$.*

($\leftarrow$) *Assume $M_T^*, s\models^*\triangle_i\phi$, if and only if for all t with $s\rightarrow_i^T t$,such that $M_T^*, t\models^*\phi$ or for all t with $s\rightarrow_i^T t$,such that $M_T^*, t\models^*\neg\phi$. To show $M, s\models\triangle_i\phi$, we must show for all t with $s\rightarrow_i t$ such that $M, t\models\phi$ or for all t with $s\rightarrow_i t$ such that $M, t\models\neg\phi$.*

Now assume that for all t with $s \to_i^T t$, $M_T^*, t \models^* \phi$. *Since* $\to_i$ *is the transitive closure of* $\to_i^T \cup \to_i^+$, *there are* $v_1 ... v_k \in T$ *such that:*

(1)$v_1 = s$,

(2)$v_k = t$,

(3)either $(v_j, v_{j+1}) \in \to_i^T$ *or* $(v_j, v_{j+1}) \in \to_i^+$, *for* $i \leq j \leq k$.

Firstly, we can show that no matter how long the path is, there is $M_T^*, t \models \triangle_i \phi$.

We now show by induction on the length of the path k. The case k=1(that is s=t) is $M_T^*, s \models^* \triangle_i \phi$, *which holds trivially. Assume the case when k=n,* $M_T^*, v_n \models^* \triangle_i \phi (1 \leq j \leq k-1)$. *It follows that* $M_T^*, v_n \models^* \triangle_i \triangle_i \phi$ *by* $(\triangle 4)$. *If k=n+1, we have to show that* $M_T^*, v_{n+1} \models^* \triangle_i \phi$. *We should consider (3):*

In the first case where $v_n \to_i^T v_{n+1}$, *for all* v_{n+1} *with* $v_n \to_i^T v_{n+1}$, *there is* $M_T^*, v_{n+1} \models^* \triangle_i \phi$ *or for all* v_{n+1} *with* $v_n \to_i^T v_{n+1}$, *there is* $M_T^*, v_{n+1} \models^* \neg \triangle_i \phi$. *But* $M_T^*, v_{n+1} \models^* \neg \triangle_i \phi$ *is impossible because if it holds, there would be* t_1, t_2 *with* $v_{n+1} \to_i^T t_1$ *and* $v_{n+1} \to_i^T t_2$, *such that* $M_T^*, t_1 \models^* \phi$ *and* $M_T^*, t_2 \models^* \neg \phi$. *But* $\{\to_i^T\}$ *is transitive. So there are* $v_n \to_i^T t_1$ *and* $v_n \to_i^T t_2$, *we would get* $M_t^*, v_n \models^* \neg \triangle_i \phi$, *contradiction. So* $M_T^*, v_{n+1} \models^* \triangle_i \phi$.

In the second case where $v_n \to_i^+ v_{n+1}$, *by axiom(*$\triangle_i \phi \to D^\triangle \phi$*), there is* $M_T^*, v_n \models^* \triangle_i \triangle_i \phi \to D^\triangle \triangle_i \phi$ *by Rule(Sub)(Sub is admissible in DNCL). So* $M_t^*, v_n \models^* D^\triangle \triangle_i \phi$. *According to the definition of* $\to_i^+$, *there is* $v_n \to_D v_{n+1}$ *in* M_T^*. *So for all* v_{n+1} *with* $v_n \to_D v_{n+1}$, *there is* $M_T^*, v_{n+1} \models^* \triangle_i \phi$ *or for all* v_{n+1} *with* $v_n \to_D v_{n+1}$, *there is* $M_T^*, v_{n+1} \models^* \neg \triangle_i \phi$. *Prove it by contradiction. If for all* v_{n+1} *with* $v_n \to_D v_{n+1}$, *there is* $M_T^*, v_{n+1} \models^* \neg \triangle_i \phi$. *According to the definition of* $\to_D$, $v_n \to_D v_{n+1}$ *iff there exists* χ *such that 1.*$\neg D^\triangle \chi \in v_n$, *2.*$\neg \triangle_i \phi \in v_{n+1}$ *implies* $\neg (D^\triangle \neg \triangle_i \phi \wedge D^\triangle (\chi \to \neg \triangle_i \phi)) \in v_n$. *Since* $D^\triangle \neg \triangle_i \phi \leftrightarrow D^\triangle \triangle_i \phi$, $D^\triangle \triangle_i \phi \in v_n$ *implies* $\neg D^\triangle \triangle_i \phi \notin v_n$ *according to the definition of maximal consistent sets. And since* $M_T^*, v_{n+1} \models^* (\chi \to \neg \triangle_i \phi)$ *for all* v_{n+1} *with* $v_n \to_D v_{n+1}$, *there is* $M_T^*, v_n \models^* D^\triangle (\chi \to \neg \triangle_i \phi)$. *So* $\neg D^\triangle (\chi \to \neg \triangle_i \phi) \notin v_n$. *This contradicts to 2. So for all* v_{n+1} *with* $v_n \to_D v_{n+1}$, *we have* $M_T^*, v_{n+1} \models^* \triangle_i \phi$.

Secondly, we can should show that no matter how long the path is, there is $M_T^*, t \models^* \phi$. *Do induction on the length k of the path.*

If k=1, there is $s \to_i^T$ *since* $\to_i^T$ *is reflexive. And we have assumed that for all t with* $s \to_i^T t$, $M_T^*, t \models^* \phi$. *Thus we have* $M_T^*, s \models^* \phi$.

The inductive hypothesis is that when k=n, there is $M_T^*, s \models^* \phi$.

If k=n+1,

In the first case where $v_n to_i^T v_{n+1}$, *we have known that* $M_T^*, v_n \models^* \phi$ *and* $M_T^*, v_n \models^* \triangle_i \phi$. *Since* $\to_i^T$ *is reflexive, we could know that for all u with* $v_n \to_i^T u$, *there is* $M_T^*, u \models \phi$.

In the second case where $v_n \to_i^+ v_{n+1}$, *we have known that* $M_T^*, v_n \models^* \triangle_i \phi$ *and the axiom* $\triangle_i \phi \to D^\triangle \phi$. *Thus we have* $M_T^*, v_n \models D^\triangle \phi$. *Because* $\to_D$ *is reflexive and* $v_n \to_D t$, *there is* $M^*, t \models \phi$.

This completes the induction. Now we have proved that for all t with $s \to t$ *in M, there is* $M, t \models \phi$. *Thus* $M, s \models \triangle_i \phi$. *Assume* $M_T^*, u \models^* \phi$, *there is* $M, s \models \triangle_i \phi$ *similarly.*

Case 2: If $\psi = D^\triangle \phi$:

$(\to)$ *Assume* $M_T^*, s \not\models^* D^\triangle \phi$. *Thus, there are* t_1 *and* t_2 *with* $s \to_D t_1$ *and* $s \to_D t_2$, *such that* $M_T^*, t_1 \models^* \phi$ *and* $M_T^*, t_2 \models^* \neg \phi$. *By IH,* $M, t_1 \models \phi$ *and* $M, t_2 \models \neg \phi$, *since* $s{\rightarrow}_D t_1$ *and* $s{\rightarrow}_D t_2$*, according to the definition of* $\rightarrow_i^+$*, there are* $s{\rightarrow}_i t_1$ *and* $s{\rightarrow}_i t_1$ *for all* $i{\in}A$*. So we get* $M, s{\not\models}D^{\triangle}\phi$.

($\leftarrow$) *Assume now that* $M, s{\not\models}D^{\triangle}\phi$*. Thus, there are states* t_1, $t_2{\in}T$ *with* $s{\rightarrow}_i t_1$, $s{\rightarrow}_i t_2$ *for each* $i{\in}A$*, such that* $M, t_1{\models}\phi$ *and* $M, t_2{\models}\neg\phi$*. By IH,* $M_T^*, t_1{\models}^*\phi$ *and* $M_T^*, t_2{\models}^*\neg\phi$*. Since* $s{\rightarrow}_i t_1$*, there is a reduced path* $P_1 = \langle v_l, \alpha, v_{l+1}, \alpha ..., \alpha, v_{l+k}\rangle$ *where* $v_l = s$*,* $v_{l+k} = t_1$ *and* α *is* i *or* $D^{\triangle}$*, for some* $i{\in}A$*. The reduced paths are a kind of paths where every adjacent nodes are in the adjacent layers. Now I will show that for any* $j{\in}A$*,* P_1 *is also a path for them from* s *to* t_1.

Suppose there is another reduced path $P_2 = \langle v_l, \beta, v_{l+1}, \beta ..., \beta, v_{l+k}\rangle$ *where* $v_l = s$*,* $v_{l+k} = t_1$ *and* β *is* $j (j \not= i)$ *or* $D^{\triangle}$*, for some* $j{\in}A$*. Since* P_1 *and* P_2 *have the same beginning node and the ending node and* M_T^* *is tree-like, any node in* M_T^* *can't have two different successors in the adjacent higher layer. Prove it by contradiction. If* $P_1 = P_2$*, that means there is at least one node in* P_2 *has two successors in the adjacent higher layer. Thus we have to say* $P_2 \not= P_1$ *and* α *is* $\rightarrow_D^T$*. By transitivity of* $\{\rightarrow_i^T | i{\in}A\}$*, we have* $s{\rightarrow}_D^T t_1$ *and* $s{\rightarrow}_D^T t_2$*. So we have* $M_T^*, s{\not\models}D^{\triangle}\phi$.

So we proved the proposition.

Theorem 2. *The logic DNCLS5 is complete with respect to the class* $\mathcal{S}5$ *of all frames.*

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